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ABSTRACT

J. A. Tomlin published a paper [1] on meeting required multicommodity network flows at minimum cost. He formulated this problem in both node-arc and arc-chain form. The node-arc linear program was attacked by the Dantzig-Wolfe decomposition principle by expressing the derived master program as convex combinations of the extreme points of the derived subprograms. In this note, it is shown that this problem is really a special case of the problem where one is attempting to meet minimum cost multicommodity flows without flow requirements on the individual commodities. Tomlin's algorithm is then modified to solve this more general problem. When this is done, the subprograms are homogeneous and the master program is a nonnegative combination of their independent solutions.

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INTRODUCTION

Consider a network $[N,A]$ with n nodes and m directed arcs joining pairs of nodes. Assign (i,j), the arc directed from i to j, a capacity $b_{ij} \geq 0$ and a cost c_{ij} per unit of flow. The c_{ij}'s are such that the total cost on any directed cycle is nonnegative. Commodity k, $k = 1, \ldots, q$, is identified by its source s_k and its sink t_k. It is required to find a multicommodity flow that satisfies the capacity constraints at minimum cost.

Tomlin [1] published a paper in which $c_{ij} \geq 0$ and a flow r_k of commodity k was required. It will be shown later that this problem is a special case of the one treated here. Other special cases treated by this more general formulation are maximizing a linear combination of the k distinct commodity flows and finding an efficient routing when the value of a routing is proportional to individual commodity flows but this value can be offset by transportation costs.

DECOMPOSITION

Let y_{ij}^k be the flow of commodity k on arc (i,j) and v_k the total flow of commodity k. Then it is required to find y_{ij}^k, v_k, and min Z such that

(1) $$Z = \sum_{(i,j)} c_{ij}\left(\sum_{k=1}^{q} y_{ij}^q\right)$$

(2) $$\sum_{k=1}^{q} y_{ij}^k \leq b_{ij} \quad \text{all } (i,j)$$

(3) $$0 = \sum_{j=1}^{n} (y_{ij}^k - y_{ji}^k) + \begin{cases} -v_k & \text{for } i = s_k \\ v_k & \text{for } i = t_k \\ 0 & \text{otherwise} \end{cases}$$

$$k = 1, \ldots, q$$

Letting A_k be the node-arc incidence matrix of the network; d_k an n vector containing -1 in the s_k position, + 1 in the t_k position, and 0 elsewhere; y_k the vector of arc flows $< y_{ij}^k >$ for commodity k; b the vector of arc capacities; c' the vector $< c_{ij} >$ of arc costs; and s a vector of slacks, (1-3) may be written as

(4) $$\begin{array}{lllllr} c'y_1 & + c'y_2 & + \ldots & + c'y_q & & = Z \min \\ Iy_1 & + Iy_2 & + \ldots & + Iy_q & + Is & = b \\ A_1y_1 + d_1v_1 & & & & & = 0 \\ & A_2y_2 + d_2v_2 & & & & = 0 \\ & & \ldots & & & \\ & & & A_qy_q + d_qv_q & & = 0 \end{array}$$

Note that solutions to $A_k y_k + d_k v_k = 0$ may be decomposed into units of flow along paths from s_k to t_k and along cycles. However, since all cycles have nonnegative cost and since elimination of flow on cycles from a feasible solution cannot yield an infeasible solution, cycle flows need not appear in an optimal solution and can be eliminated from consideration. Thus, let $W_k = \{ w_{k1}, \ldots, w_{kN_k} \}$ be the set of solutions corresponding to one unit of flow on a directed path from s_k to t_k. W_k is then a set of points that span the set of solutions to $A_k y_k + d_k v_k = 0$ which contain no cycles. Then, applying the Dantzig-Wolfe decomposition principle [2] one may write y_k as a nonnegative combination of the elements of W_k and (4) becomes

$$(5)\qquad \begin{aligned} &\sum_{j=1}^{N_1} \lambda_{1j} \, (c' w_{ij}) + \ldots + \sum_{j=1}^{N_q} \lambda_{qj} \, (c' w_{qj}) = Z \text{ min} \\ &\sum_{j=1}^{N_1} \lambda_{1j} \, (I \, w_{ij}) + \ldots + \sum_{j=1}^{N_q} \lambda_{qj} \, (I \, w_{qj}) + Is = b \\ &\lambda_{kj} \geq 0 \end{aligned}$$

The number of variables in (5) is of course too large to enumerate. However, suppose we have a basic feasible solution and let π_{ij} be the corresponding simplex multiplier for the row containing b_{ij} on the right-hand side and $\pi' = < \pi_{ij} >$. If $\pi_{ij} > 0$, then s_{ij} may be introduced into the basis to give an improved basic solution. If all $\pi_{ij} \leq 0$, then any solution with

$$(6)\qquad c' w_{kj} - \pi' \, I \, w_{kj} < 0$$

will yield an improvement. If no w_{kj} satisfies (6), then the current solution is optimal. If arc (i,j) is assigned a cost of $c_{ij} - \pi_{ij}$, then the left-hand side of (6) is equal to the total cost of the path represented by w_{kj}. Thus the search for a solution satisfying (6) may be found by finding the shortest path between s_k and t_k for k = i, ..., q. Efficient methods for finding shortest chains are given in [3-5]. Many of them require that the network contain no negative cycles. This is assured when all $\pi_{ij} \leq 0$ due to our initial restriction of the c_{ij}s.

The phase I procedure for finding a starting feasible basis to initiate the algorithm is accomplished by setting Is = b.

FLOW REQUIREMENTS

Tomlin [1] treats the problem where it was required to minimize total network cost subject to the restriction that the total flow of commodity k be equal to r_k. Furthermore, it was assumed that all $c_{ij} \geq 0$. This can be treated as a special case of the problem treated in this paper. For each k one merely attaches an artificial node and directs an artificial arc from it to s_k (or alternatively to it from t_k). This artificial arc is assigned a capacity of r_k and a very large negative cost. Since none of these artificial arcs belong to cycles, our initial restriction on the c_{ij}s will hold.

The node-arc formulation in (1) differs only slightly from ours. Specifically, (4) is replaced by

$$
\begin{array}{lllllll}
(7) & c'y_1 & + \; c'y_2 & + \; \cdots & + \; c'y_q & & = Z \text{ min} \\
& Iy_1 & + \; Iy_2 & + \; \cdots & + \; Iy_q & + \; Is & = b \\
& A_1y_1 & & & & & = d_1 \\
& & A_2y_2 & & & & = d_2 \\
& & & & & \cdots & \\
& & & A_qy_q & & & = d_q \\
& y_k \geq 0 \quad \text{all } k & & & & &
\end{array}
$$

Here d_k is the vector with an r_k in the s_k position, a $-r_k$ in the t_k position, and zero everywhere else; all other variables are as defined before.

This problem is also treated by decomposition. However, the subprograms, $A_qy_q = d_q$, are no longer homogeneous and consequently, instead

of looking for a nonnegative combination of independent solutions to the subprograms, we look for convex combinations of their extreme points. Thus letting $W_k = \{w_{k1}, \ldots, w_{kN_k}\}$ be the extreme points of $A_k y_k = d_k$, (5) is replaced by

$$
(8) \qquad
\begin{array}{lllll}
\sum_{j=1}^{N_1} \lambda_{ij} (c' w_{ij}) & + \ldots + & \sum_{j=1}^{N_q} \lambda_{qj} (c' w_{qj}) & & = Z \text{ min} \\
\sum_{j=1}^{N_1} \lambda_{ij} (I w_{ij}) & + \ldots + & \sum_{j=1}^{N_q} \lambda_{qj} (I w_{qj}) & + Is & = b \\
\sum_{j=1}^{N_1} \lambda_{ij} & & \ldots & & = 1 \\
& & & \sum_{j=1}^{n_q} \lambda_{qj} & = 1 \\
& \lambda_{kj} \geq 0 & & &
\end{array}
$$

The extreme points of these subprograms are also paths and hence the W_k defined here is identical to that defined in the last section. In form, (8) differs from (5) only in that one must add a convexity constraint on the λ_{kj} for each k. However, if (5) is used, one must also add constraints reflecting capacities on the added artificial arcs (i.e. the b vector in (5) is larger than that in (8) by the number of commodities) which differ only slightly from the convexity constraints in (8). Thus for this special case the formulations are almost identical.

ARC-CHAIN FORMULATION

In [1] the minimum cost multicommodity problem was also formulated as a linear program in terms of its arc chain incidence matrix. Specifically, rows correspond to arcs and columns to chains. The program was then solved by a simple extension of a method due to Ford and Fulkerson [6] which has since been extended by Wollmer [7]. It was shown that this method turned out to be equivalent to applying decomposition to the node-arc program.

In this paper, we will not go into the details of this method other than to comment that if the more general problem of this paper is formulated in terms of its arc-chain incidence matrix, it may also be solved by a similar extension to the method proposed in [6] and that this method is also equivalent to applying decomposition to the node-arc linear program.

CONCLUDING REMARKS

It has been shown that the problem treated in [1] is a special case of that treated here. As other special cases, the problem treated here includes (i) maximizing a linear combination of the individual commodity flows and (ii) finding an efficient routing that takes into consideration the value of both the individual commodity flows and the transportation costs involved. The former of these is accomplished by attaching, for each commodity, and artificial node and an artificial arc directed from it to the source. The artificial arcs are given infinite capacities and costs whose negatives are proportional to the linear coefficients of the commodity flows in the linear function that is to be maximized. For the latter problem, one also adds these same artificial nodes and arcs, the new arcs having infinite capacity. The cost on the artificial arc for commodity k is the negative of the value of a unit of flow of commodity k. Thus the scope of problems treated in [1] may be significantly increased by relatively small changes in the problem formulation and algorithms.

REFERENCES

1. Tomlin, J. A., "Minimum Cost Multicommodity Network Flows," Operations Research, Vol. 14, 1966, pp. 45-51.

2. Dantzig, G. B., and P. Wolfe, "The Decomposition Algorithm for Linear Programs," Econometrica 29, 1969, pp. 767-778.

3. Ford, L. R. and D. R. Fulkerson, "Flows in Networks," Princeton University Press, Princeton, New Jersey, 1963.

4. Hu, T. C., "Revised Matrix Algorithms for Shortest Paths," SIAM J. on Appl. Math., Vol. 15, No. 1., January 1967, pp. 207-218.

5. Dreyfus, S. E., An Appraisal of Some Shortest-Path Algorithms, The RAND Corporation, RM-5431-PR, September 1968.

6. Ford, L. R. and D. R. Fulkerson, "A Suggested Computation for Maximal Multicommodity Network Flows," Management Science, 5 (1958), pp. 97-101.

7. Wollmer, R. D., Generalized Minimum Cost Multicommodity Network Flows, unpublished work.